GRADE 10 ANALYTICAL GEOMETRY

CHAPTER 8 ANALYTICAL GEOMETRY

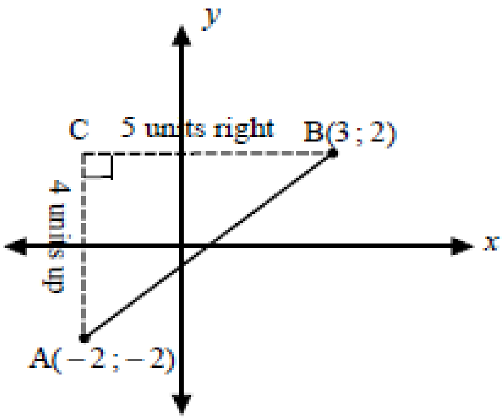
Analytical Geometry is the study of Geometry, using the Cartesian plane. It is an algebraic approach to the study of Geometry. In this chapter, we will address the following concepts:

- The **distance** between two points (length of a line segment).
- The **midpoint** of a line segment.
- The **gradient** of a line.

THE DISTANCE BETWEEN TWO POINTS (LENGTH OF A LINE SEGMENT)

Suppose that we wish to calculate the length of line segment AB, with endpoints $A(-2\,;-2)$ and $B(3\,;2)$.

Consider the diagram below. The movement from A to B has been indicated. A right-angled triangle ABC is formed, with lengths $AC = 4$ (4 units up) and $CB = 5$ (5 units right).



The theorem of Pythgoras can now be used to calculate the length of AB:

$\therefore AB^2 = BC^2 + AC^2$

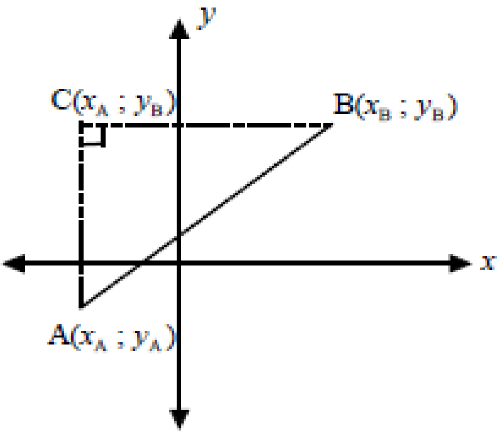
$\therefore AB^2 = (\text{horizontal movement})^2 + (\text{vertical movement})^2$

$\therefore AB^2 = 5^2 + 4^2 = 41$

$\therefore AB = \sqrt{41} = 6,40$

We can generalise this concept to create what is known as the "*Distance Formula*".

Consider any two points $A(x_A\,;y_A)$ and $B(x_B\,;y_B)$.



From the diagram alongside:

$BC = \text{horizontal movement} = x_B - x_A$

$AC = \text{vertical movement} = y_B - y_A$

$\therefore AB^2 = BC^2 + AC^2$ Pythagoras

$\therefore AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

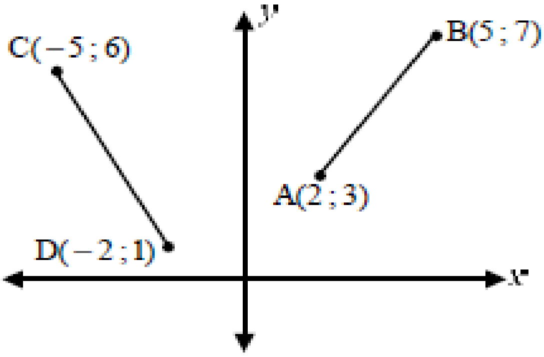
$\therefore AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

The formula to calculate the length of a line segment between points A and B, with $A(x_A\,;y_A)$ and $B(x_B\,;y_B)$, is:

$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$ or $AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

EXAMPLE 1

Calculate the lengths of line segments AB and CD in the given diagram.



Solutions

(a)

$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

$AB^2 = (5-2)^2 + (7-3)^2$

$AB^2 = (3)^2 + (4)^2$

$AB^2 = 25$

$AB = \sqrt{25} = 5$ units

(b)

$CD^2 = (x_D - x_C)^2 + (y_D - y_C)^2$

$CD^2 = (-2-(-5))^2 + (1-6)^2$

$CD^2 = (3)^2 + (-5)^2$

$CD^2 = 34$

$CD = \sqrt{34} = 5,83$ units

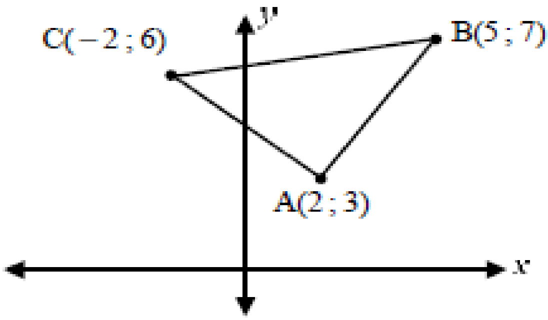
APPLICATIONS OF THE DISTANCE FORMULA

EXAMPLE 2

In the diagram, the vertices of ΔABC are $A(2;3)$, $B(5;7)$ and $C(-2;6)$.

(a) Show that ΔABC is an isosceles triangle.

(b) Calculate the perimeter of ΔABC correct to one decimal place.



Solutions

(a) We can show that ΔABC is an isosceles triangle by proving two sides equal. From the diagram above, the obvious choice is to prove that $AB = AC$.

$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

$AB^2 = (5-2)^2 + (7-3)^2$

$AB^2 = 25$

$AB = \sqrt{25} = 5$ units

$AC^2 = (x_C - x_A)^2 + (y_C - y_A)^2$

$AC^2 = (-2-2)^2 + (6-3)^2$

$AC^2 = 25$

$AC = \sqrt{25} = 5$ units

$\therefore AB = AC$

(b) The perimeter of ΔABC is the sum of its three sides:

$BC^2 = (x_C - x_B)^2 + (y_C - y_B)^2$

$BC^2 = (-2-5)^2 + (6-7)^2$

$BC^2 = 50$

$BC = \sqrt{50}$

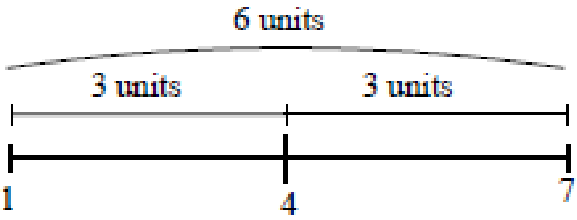
$\text{Perimeter} = AB + AC + BC$

$\therefore \text{Perimeter} = 5 + 5 + \sqrt{50} = 17,071... = 17,1$ units

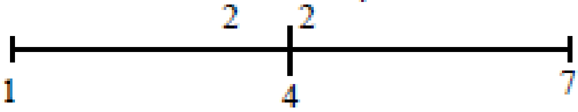
THE MIDPOINT OF A LINE SEGMENT

The midpoint of a line segment is the halfway mark on the line segment. Consider the numbers 1 and 7 for example. Halfway between 1 and 7 is 4. How do we get to 4? One way is to take the difference between 1 and 7 (which is 6), half that (which is 3), and then add this 3 to the 1 to get to 4. (You could also subtract 3 from 7 to get to 4).

The diagram on the right illustrates this approach.



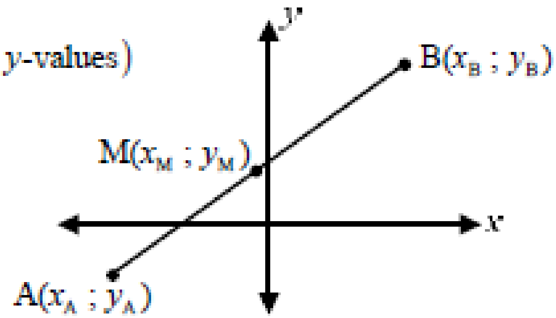
A quicker approach is to **add the two end values and then divide the answer by 2.**



This is basically working out the **average of the two end values.**
If we apply this concept to a line segment joining two points on the Cartesian plane, we can easily find the midpoint of the line segment by calculating the average of the x-values and the average of the y-values.

$$M(x_M\,;\,y_M)=M(\text{average of the } x\text{-values}\,;\,\text{average of the } y\text{-values})$$

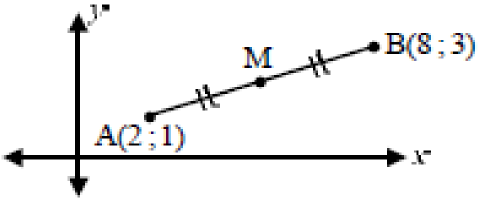
$$\therefore M(x_M\,;\,y_M)=M\left(\frac{x_A+x_B}{2}\,;\,\frac{y_A+y_B}{2}\right)$$



The formula to calculate the midpoint of a line segment between points A and B, with $A(x_A\,;\,y_A)$ and $B(x_B\,;\,y_B)$, is: $M(x_M\,;\,y_M)=M\left(\frac{x_A+x_B}{2}\,;\,\frac{y_A+y_B}{2}\right)$

EXAMPLE 5

Determine the coordinates of M, if M is the midpoint of line segment AB, where $A(2;1)$ and $B(8;3)$.

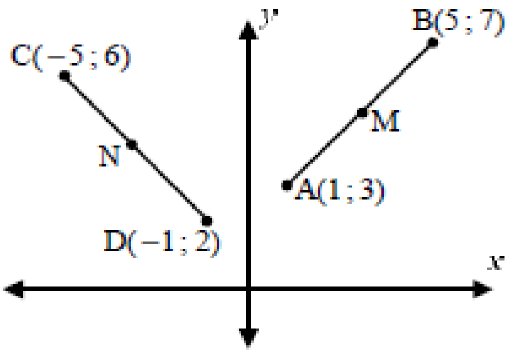


Solution

$$M\left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)$$
$$= M\left(\frac{2+8}{2}; \frac{1+3}{2}\right)$$
$$= M(5;2)$$

EXAMPLE 6

Calculate the midpoints of line segments AB and CD in the given sketch.



Solutions

Midpoint of AB is M:

$$M(x_M; y_M) = M\left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)$$
$$\therefore M\left(\frac{1+5}{2}; \frac{3+7}{2}\right)$$
$$\therefore M(3;5)$$

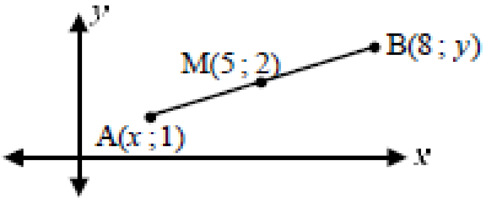
Midpoint of CD is N:

$$N(x_N; y_N) = N\left(\frac{x_C + x_D}{2}; \frac{y_C + y_D}{2}\right)$$
$$\therefore N\left(\frac{-5+(-1)}{2}; \frac{6+2}{2}\right)$$
$$\therefore N(-3;4)$$

APPLICATIONS OF THE MIDPOINT FORMULA

EXAMPLE 7

Determine the values of x and y if $M(5;2)$ is the midpoint of the line segment joining the points $A(x;1)$ and $B(8;y)$.



Solution

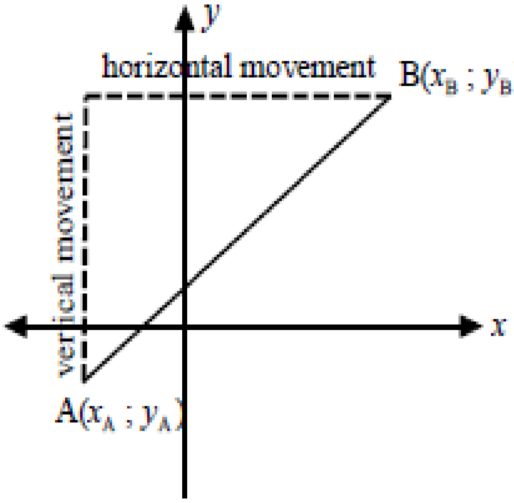
$x_M = \frac{x_A + x_B}{2}$	$y_M = \frac{y_A + y_B}{2}$
$\therefore 5 = \frac{x+8}{2}$	$\therefore 2 = \frac{1+y}{2}$
$\therefore 10 = x + 8$	$\therefore 4 = 1 + y$
$\therefore x = 2$	$\therefore y = 3$

THE GRADIENT OF A LINE SEGMENT

Gradient (or **slope**) measures the **steepness** and **direction** of a line. A line can either slant up (gradient is positive), slant down (gradient is negative), be horizontal (gradient is zero) or be vertical (gradient is undefined). The symbol used for gradient is m.

In Grade 9 we calculated the gradient of a line using the concept of "*rise over run*". In other words: $\text{Gradient} = \dfrac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}} = \dfrac{\text{vertical movement}}{\text{horizontal movement}}$.

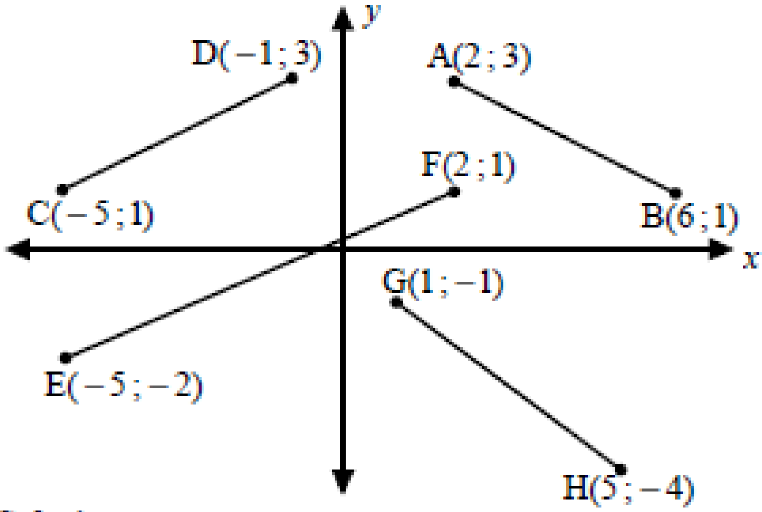
This can easily be translated into a formula. For any two points $A(x_A\,;y_A)$ and $B(x_B\,;y_B)$:
Vertical movement $= y_B - y_A$ and Horizontal movement $= x_B - x_A$.



A formula to calculate the gradient of a line joining two points A and B, witl $A(x_A\,;y_A)$ and $B(x_B\,;y_B)$, is: $m_{AB} = \dfrac{y_B - y_A}{x_B - x_A}$

EXAMPLE 9

Calculate the gradients of the following lines using the formula for gradient.



Solutions

$m_{AB} = \dfrac{y_B - y_A}{x_B - x_A} = \dfrac{1-3}{6-2} = \dfrac{-2}{4} = -\dfrac{1}{2}$ (slopes down from left to right)

$m_{CD} = \dfrac{y_D - y_C}{x_D - x_C} = \dfrac{3-1}{-1-(-5)} = \dfrac{2}{4} = \dfrac{1}{2}$ (slopes up from left to right)

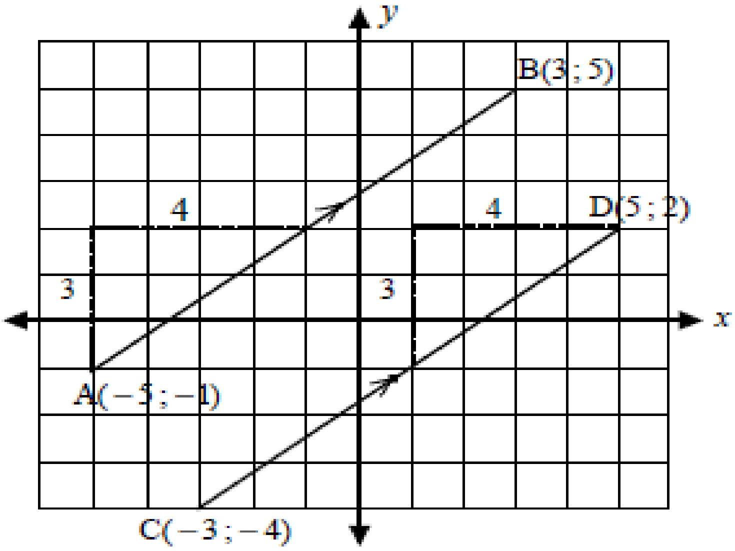
$m_{EF} = \dfrac{y_F - y_E}{x_F - x_E} = \dfrac{1-(-2)}{2-(-5)} = \dfrac{3}{7}$ (slopes up from left to right)

$m_{GH} = \dfrac{y_H - y_G}{x_H - x_G} = \dfrac{-4-(-1)}{5-1} = \dfrac{-3}{4} = -\dfrac{3}{4}$ (slopes down from left to right)

APPLICATIONS OF GRADIENT

Parallel lines

Parallel lines slope in the exactly the same direction and will therefore never intersect. Differently stated: **Lines that are parallel have <u>equal</u> gradients.**



For any pair of parallel lines AB and CD: $m_{AB} = m_{CD}$

EXAMPLE 10

Given are the points $A(-1\,;5)$, $B(-2\,;3)$, $C(9\,;10)$ and $D(5\,;2)$. Show that AB||CD.

Solution

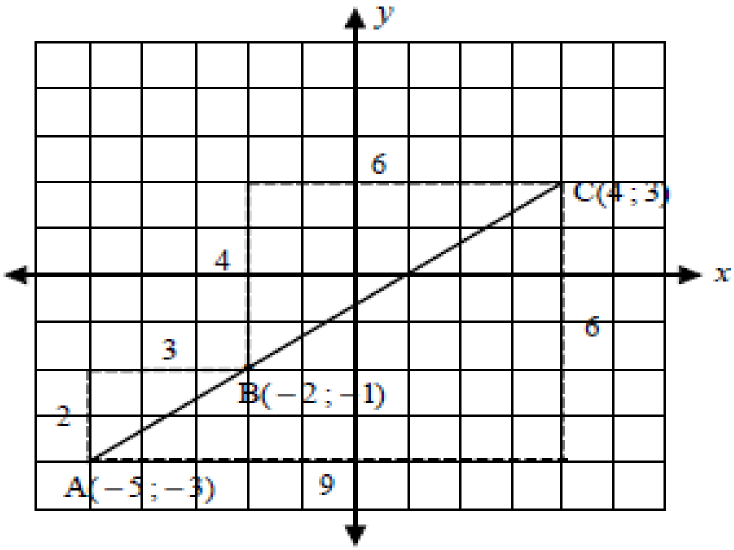
$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3-5}{-2-(-1)} = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$ $\quad\quad$ $m_{CD} = \frac{y_D - y_C}{x_D - x_C} = \frac{2-10}{5-9} = \frac{-8}{-4} = 2$

$\therefore m_{AB} = m_{CD}$

$\therefore$ AB||CD

Collinear points

Points are said to be **collinear** when they lie on the same line. Refer to the diagram. A, B and C lie on the same line and are therefore collinear. This implies that the **gradients between each pair of points are the same.**



When points A, B and C are collinear: $m_{AB} = m_{AC} = m_{BC}$
In other words: $m_{AB} = m_{AC}$ and $m_{AB} = m_{BC}$ and $m_{AC} = m_{BC}$

EXAMPLE 11

Show that the points A, B and C are collinear if the coordinates of the points are:
$A(2\,;-2)$, $B(1\,;1)$ and $C(-1\,;7)$.

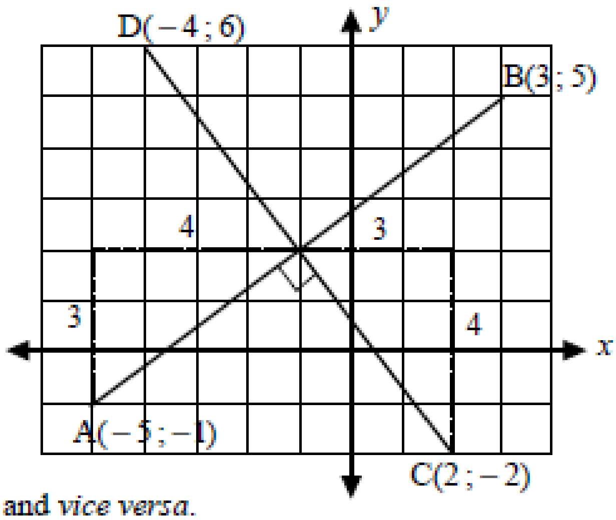
Solution

We will consider the gradients of AB and BC, but any other pair could have been used.

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-(-2)}{1-2} = \frac{3}{-1} = -3 \quad \text{and} \quad m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{7-1}{-1-1} = \frac{6}{-2} = -3$$

$\therefore m_{AB} = m_{BC}$
Therefore A, B and C are collinear.

Perpendicular lines



Perpendicular lines intersect at a 90° angle. The gradients of perpendicular lines have a particular property.
Consider the diagram on the right.
Firstly:
The gradients of the lines have opposite signs. AB has a positive gradient whereas CD has a negative gradient.
Secondly:
The gradients (ignoring signs) are reciprocals of one another. In other words, the horizontal movement of AB is the vertical movement of CD and *vice versa*.

This can be summarised by the following relationship: The product of the gradients of AB and CD will equal -1 when AB is perpendicular to CD.

For any pair of *perpendicular lines* AB and CD: $m_{AB} \times m_{CD} = -1$

EXAMPLE 12

Given are the points $A(3\,;-3)$, $B(6\,;-7)$, $C(-5\,;0)$ and $D(-1\,;3)$.
Show that AB is perpendicular to CD.

Solution

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{-7-(-3)}{6-3} = \frac{-7+3}{3} = \frac{-4}{3}$$

$$m_{CD} = \frac{y_D - y_C}{x_D - x_C} = \frac{3-0}{-1-(-5)} = \frac{3}{-1+5} = \frac{3}{4}$$

$$\therefore m_{AB} \times m_{CD} = \frac{-4}{3} \times \frac{3}{4} = -1$$

$\therefore AB \perp CD$

(c) Determine whether line segments AB and CD are parallel, perpendicular or neither, in each of the following cases.

(1) $A(-1;-3)$, $B(2;1)$ and $C(4;-1)$, $D(7;3)$.

(2) $A(1;-3)$, $B(2;1)$ and $C(4;-1)$, $D(7;3)$

(3) $A(1;-3)$, $B(2;1)$ and $C(-3;1)$, $D(1;0)$

(d) (1) Line segment AB is parallel to line segment CD. $A(-5;-1)$ and $B(-3;a)$ are points on AB. $C(-4;-3)$ and $D(-1;3)$ are points on CD. Calculate the value of a.

(2) Line segment AB is perpendicular to line segment CD. $A(-5;2)$ and $B(b;-1)$ are points on AB. $C(-4;-3)$ and $D(-1;3)$ are points on CD. Calculate the value of b.

(e) Show that points F, R and N are collinear if $F(3;2)$, $R(4;-2)$ and $N(7;-14)$.

(f) Calculate the value(s) of x if $R(x;4)$, $U(x-1;x+4)$ and $N(0;13)$, are collinear.

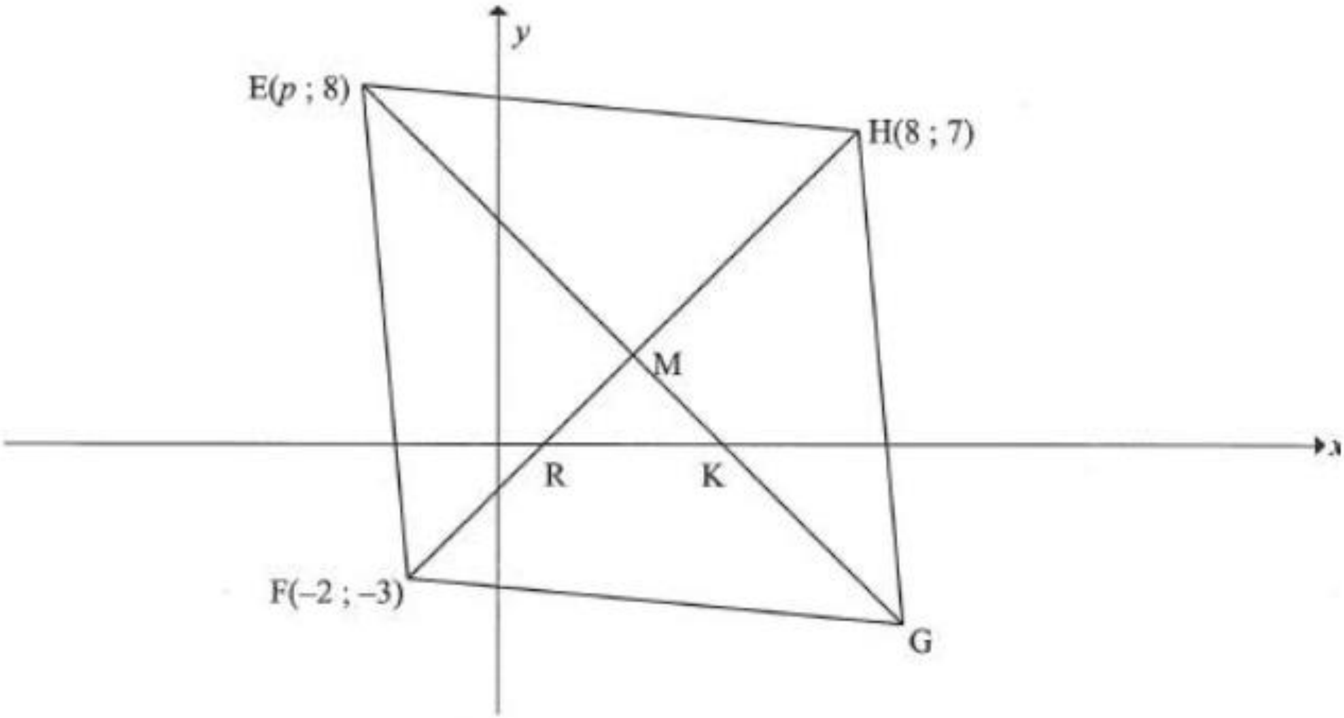
(g) $A(3;4)$, $B(-1;7)$, $C(x;-1)$ and $D(1;8)$ are points on the Cartesian plane. Calculate the value of x in each case if:

(1) $AB \parallel CD$ (2) $AB \perp CD$ (3) B, C and D are collinear

Past examination questions on Analytical Geometry

QUESTION 3

In the diagram below, E(p; 8), F(–2; –3), G and H(8; 7) are vertices of rhombus EFGH. The diagonals EG and HF intersect at M and cut the x-axis at K and R respectively.



3.1 Calculate the:

3.1.1 Coordinates of M (2)

3.1.2 Gradient of FH (2)

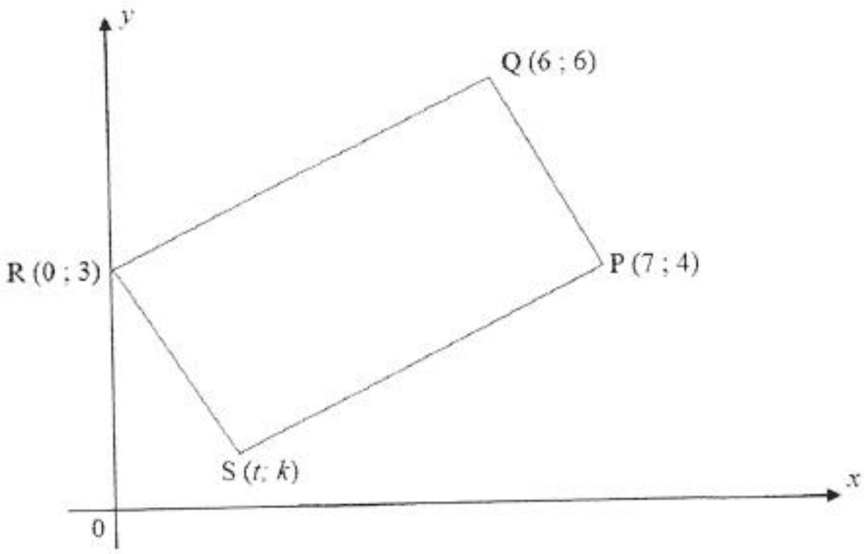
3.1.3 Size of $M\hat{K}R$ (4)

3.2 Use the properties of a rhombus to calculate the value of p. (4)

3.3 Calculate the coordinates of G. (2)

QUESTION 3

In the diagram below, P(7 ; 4), Q(6 ; 6), R(0 ; 3) and S(t ; k) are the vertices of quadrilateral PQRS.



3.1 Calculate the length of PQ. Leave your answer in surd form. (2)

3.2 If $T\left(\frac{7}{2};\frac{7}{2}\right)$ is the midpoint of QS, determine the coordinates of S. (3)

3.3 If the coordinates of S are (1 ; 1), show that PR = QS. (2)

3.4 Show that $QR \perp RS$. (4)

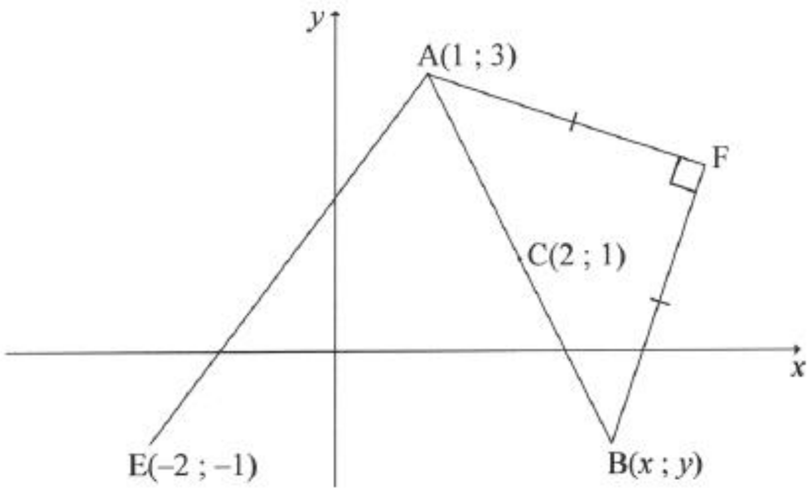
3.5 Hence, what type of special quadrilateral is PQRS? Motivate your answer. (2)

3.6 Calculate the size of $R\hat{S}Q$. (3)

[16]

QUESTION 2

In the diagram below, A(1 ; 3), B(x ; y) and E(–2 ; –1) are points on a Cartesian plane. C(2 ; 1) is the midpoint of AB. Also, F is a point such that AF = FB and AF ⊥ FB.



2.1 Determine the:

2.1.1 Length of AE (2)

2.1.2 Gradient of AC (2)

2.1.3 Coordinates of B (3)

2.2 BE is joined to form a special quadrilateral AFBE. Name the special quadrilateral AFBE. Give full justification for your answer. (3)

2.3 Calculate the area of ΔAFB. (5)

[15]